Motivational Problems on Polar Coordinates and Complex Numbers

1. The point $P = (-5,8)$ is in the second quadrant. You are used to describing it by using the *rectangular coordinates* –5 and 8. It is also possible to accurately describe the location of P by using a different pair of coordinates: its *distance from the origin* and an *angle in standard position*. Calculate two such numbers. These are called *polar coordinates*. There is more than one correct answer.

2. Polar coordinates for a point P in the xy-plane consist of two numbers r and ϑ , where r is the distance from P to the origin O, and ϑ is the size of an angle in standard position that has OP as its terminal ray. Find polar coordinates for each of the following points:

(a) (0,1) (b) (–1,1) (c) (4,–3) (d) (1,7) (e) (–1,–7)

3. Do problems 2-5 on p.400 for more practice in changing rectangular coordinates to polar coordinates.

4. Given the rectangular coordinates (x, y) how would you right write the polar coordinates. Write a brief description and draw a diagram to illustrate your answer.

5. How would you interpret the polar coordinates $(-5,45°)$. What quadrant is this point in? Explain why it appears to be where it is.

6. Points P and Q on the unit circle are reflected images of each other, using the line $y = x$ as a mirror. Suppose that P is described by the acute polar angle ϑ ; what polar angle describes Q? In terms of ϑ, what are the rectangular coordinates of P? Find two different ways of writing the rectangular coordinates for Q.

7. Points P and Q on the unit circle are reflected images of each other, using the y-axis as a mirror. Suppose that P is described by the polar angle ϑ ; what polar angle describes Q? In terms of ϑ , what are the rectangular coordinates of P? Find two different ways of writing the rectangular coordinates for Q.

8. Points P and Q on the unit circle are reflected images of each other, using the x-axis as a mirror. Suppose that P is described by the polar angle ϑ; what polar angle describes Q? In terms of ϑ , what are the rectangular coordinates of P? Find two different ways of writing the rectangular coordinates for Q.

9. Find polar coordinates for the point described by $x = 4$ and $y = -7$.

10. Describe the configuration of all points whose polar coordinate r is 3. Describe the configuration of all points whose polar coordinate ϑ is 110.

11. Convert the polar coordinates $r = 8$ and $\vartheta = 150$ to equivalent Cartesian coordinates x and y.

12. Generalize again: Given polar coordinates r and ϑ for a point, how do you calculate the Cartesian coordinates x and y for the same point? Do problems 7-10 on p.400 for more practice if you need to.

13. How do you write the line y=x in polar form? (Hint: think of the angle that is made with the x axis).

14. In general, write a formula for finding the equation of a line with slope m, in polar form.

15. You may have noticed that under the MODE menu of your calculator you a choice that's called "Pol". This is the Polar Graphing Mode on the calculator. When you choose this mode, the Y= menu will have r= instead of y's now. When you type equations in here, they will be graphing in polar mode. Type in the equation r=5. What do you see? What is the difference between the WINDOW menu between the function mode and the Polar mode on your calculator?

16. With a partner, do exercise #1 (Computer Exercises) on p. 402 in your textbook. Come to class with some generalizations about polar graphs.

17. Graph the following Polar Graphs, and comment on any predictions you were able to make about the graphs.

    a. $r = \cos(2\vartheta)$ b. $r = \sin(3\vartheta)$ c. $r = \sin(5\vartheta)$

18. Before trying to graph it in your calculator, make an attempt at predicting what the graph of $r = \vartheta$ would look like. Try it.

19. *Spirals* are fundamental curves, but awkward to describe using only the Cartesian coordinates *x* and *y*. The example of the logarithmic spiral, on the other hand, is easily described with polar coordinates – all its points fit the equation $r = 2^{\frac{\vartheta}{360}}$ (using degree mode). Choose three specific points from a graph on your calculator and make calculations that confirm this. What range of ϑ values does the graph represent?

20. How does the logarithmic spiral compare to the *Spiral of Archimedes* from #18? Are they the same?

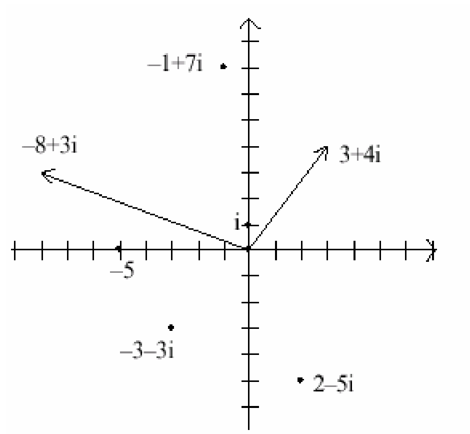
21. Many quadratic equations do not have real solutions. The simplest example is $x^2 + 1 = 0$. Rather than continuing to ignore such problems, let us do something about them. The traditional approach is to let *i* stand for a number that has the property $i^2 = -1$. The solutions to $x^2 + 1 = 0$ are therefore $x = i$ and $x = -i$. Verify this, and also show that the solutions to $x^2 - 4x + 5 = 0$ are $x = 2 + i$ and $x = 2 - i$.

22. Find the solutions to $x^2 - 6x + 13 = 0$, expressing them in the same $a + bi$ form as in #21.

23. (not a question, just information, read on!) Numbers of the form $a + bi$ are called *complex numbers*. They are often called *imaginary numbers*, but this is inaccurate, for ordinary real numbers are included among them (for example, 3 is the same as $3 + 0i$). Strictly speaking, the number *i* is called the *imaginary unit*, and *bi* is called *pure imaginary* if *b* is a nonzero real number.

24. (BONUS) Let the focal point *F* be at the origin, the horizontal line $y = -2$ be the directrix, and $P = (r, \theta)$ be equidistant from the focus and the directrix. *Using the polar variables r and* θ, write an equation that says that the distance from *P* to the directrix equals the distance from *P* to F. The configuration of all such *P* is a familiar curve; make a rough sketch of it. Then rearrange your equation so that it becomes put your calculator into polar mode, and graph this familiar curve. On which polar ray does no point appear?

25. Whatever these complex numbers may represent, it is important to be able to visualize them. Here is how to do it: The number $a + bi$ is matched with the point (a, b), or with the vector [a, b] that points from the origin to *(*a, b). Points (0, y) on the y-axis are thereby matched with pure imaginary numbers $0 + yi$, so the y-axis is sometimes called the *imaginary axi*s. The x-axis is called the *real axis* because its points *(*x, 0) are matched with real numbers $x\,i + 0$. The real-number line can thus be thought of as a subset of the *comple*x-*number plan*e. With all that said, now plot the following complex numbers: 1 + i, –5i, 1 3 + *i*



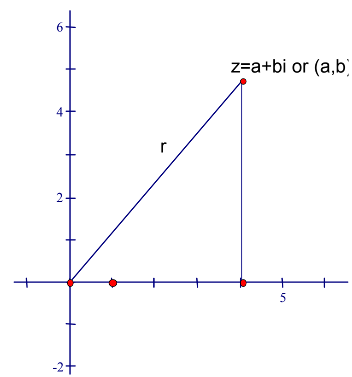
26. Because complex numbers have two components, which are usually referred to as *x* and y, it is traditional to use another letter to name complex numbers, as in $z = x + yi$. The components of $z = x + yi$ are usually called the *real part* and the *imaginary part* of z. Notice that *i* is not included in the imaginary part; for example, the imaginary part of 3 – *4i* is –4. Even the imaginary part of a complex number is a real number.
    a. What do we call a complex number whose imaginary part is 0?
    b. What do we call a complex number whose real part is 0?
    c. Describe the configuration of complex numbers whose real parts are all 2.
    d. Describe the configuration of complex numbers whose real and imaginary parts are the same.

27. Because complex numbers are naturally identified with vectors in the plane, it makes sense to talk about their *magnitude*s. Recall that the "absolute value" notation is commonly used to indicate the length of a vector, or its magnitude. Calculate a) $|3+4i|$ b) $|1-i|$ c) $|i|$

27. Describe the configuration of all complex numbers whose magnitude is 5. Give examples.

28. Perform the following arithmetic calculations, remembering the unusual fact $i^2 = -1$. Put your answers into the standard $a + bi$ form.

(a) add 3 – *i* and 2 + *3i* (b) multiply 3 + *4i* by *i* (c) multiply 3 + *i* by itself
How does your answer to part (a) relate to the vector nature of complex numbers?



29. Suppose you wanted to write a complex number $z = a + bi$, in a polar form. In other words, you wanted to write z in terms of a radius, r, and an angle θ. Use the diagram below, to find an alternate way to write the complex number z. (Hint: recall some right triangle

θ

trigonometry!)

30. Write the complex numbers below in the polar form:
    a. $z = 6 + 8i$
    b. $z = -4 - 4i$
    c. $z = -5\sqrt{3} - 5i$
    d. $z = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

31. The useful abbreviation $cis\theta$ stands for the complex number $r\cos\theta + ir\sin\theta$. Explain the abbreviation *ci*s, then write all of the complex numbers from #30 in the abbreviated form $rcis\theta$.

32. What is $i^{1234567890}$?

33. Show that the product of the two non-real numbers 3 + *2i* and 3 – *2i* is a positive real number. Assuming that *a* and *b* denote real numbers, explain why the product *(a* + *b*i)(*a* – *b*i) is always a nonnegative real number.

34. Find the angle formed by the complex numbers 3 + *4i* and –5 + 1*2i* (considered as vectors placed tail to tail).

35. Write 3 + *4i* and 1 + *i* in polar form. Then calculate the product of 3 + *4i* and 1 + i, and write this complex number in polar form, too. Do you notice anything remarkable?

36. You are familiar with Cartesian graph paper (what you normally use, also called rectangular), which is useful for problems expressed in Cartesian coordinates. Draw what you think a sheet of *polar graph paper* would look like.

37. The product of the complex numbers cis(35) and cis(21) can itself be written in the form cis(θ) . What is θ ? What is the product of 4cis(35) and 3cis(21)?

38. Write $(1+i)^5$ in standard *a* + *bi* form.

39. Write $(2+i)^2$, $(2+i)^3$ and $(2+i)^4$ in *a* + *bi* form. Graph these three complex numbers along with 2 + i. Now write all four numbers in polar form. What patterns do you notice?

40. A truly remarkable property of complex multiplication is the angle-addition identity cis *(a)cis(b) =cis(a* + b). Use the definition of $cis\theta$ and some familiar trig identities to prove this!

41. Using what you've just proven, what would $(2cis45)^2$ be equal to?

42. Generalize about the statement $(rcis\theta)^n$. Justify your statement.

43. Given a complex number $a + bi$, its *conjugate* is the number $a - bi$. What geometric transformation is being applied? What happens when a complex number and its conjugate are multiplied? What happens when a complex number and its conjugate are added?

44. To practice using DeMoivre's Theorem to find powers of complex numbers do some problems on p. 410/ #2,3(don't worry about the argand diagram part). #5, #8

45. (Continuation) Show that the expression $\frac{1}{1+i}$ can be written in $a + bi$ form. In other words, show that the *reciprocal of a complex number* is also a complex number.

46. BONUS You may have noticed that the identity cis $(a)cis(b) = cis(a + b)$ is in exactly the same form as another familiar rule you have learned about. What rule? By the way, there is a rule for cis that covers *division* in the same way that the above rule covers multiplication. Discover the rule, test it on some examples, then use it to find the polar form of $\frac{1}{cis\theta}$.

47. Express all the solutions to the following equations in $a + bi$ form:

48. Remembering that fractional exponents mean taking roots, rewrite DeMoivre's theorem to find roots of complex numbers instead of powers. Then find the cube roots of $8i$ (see p. 412 in text for help!)

49. The quadratic equation $2z^2 + 2iz - 5 = 0$ has two solutions. Find them.

50. Show that the quotient $\frac{7+2i}{2+i}$ can be simplified to $a + bi$ form.

51. Find both square roots of i and express them in polar form and in Cartesian form.